NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
25 Waverly Place, New York 3, N. Y.

COPY NO.

IMM-NYU 225
FEBRUARY 1956

NEW YORK UNIVERSITY
INSTITUTE OF
MATHEMATICAL SCIENCES

A Mixed Boundary Potential Problem Arising From a Problem in Ship Motion

MEYER JORDAN

PREPARED UNDER
CONTRACT Nonr-285(06)
WITH THE
OFFICE OF NAVAL RESEARCH

REPRODUCTION IN WHOLE OR IN PART
IS PERMITTED FOR ANY PURPOSE
OF THE UNITED STATES GOVERNMENT.

IMM-NYU 225
February 1956

A MIXED BOUNDARY POTENTIAL PROBLEM ARISING FROM A PROBLEM IN SHIP MOTION

BY

MEYER JORDAN

This report represents results obtained at the Institute of Mathematical Sciences, New York University, under the auspices of Contract No. Nonr-285(06) with the Office of Naval Research.

New York, 1956

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

Contents

		page
1.	Introduction	1
2.	Homogeneous Case	8
3.	Inhomogeneous Case	15
4.	Summary	26
5.	Bibliography	30

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

1540

1. Introduction

The problem considered here arose in a paper by Stoker and Peters[10]* on the motion of a thin ship. In this paper the problem is set up using the mathematical theory of hydrodynamics of a nonturbulent perfect fluid, and linearized by a formal perturbation procedure. The motion of the ship is assumed to consist of small oscillations relative to a uniform translation.

For convenience in formulating the problem, three coordinate systems are introduced. The (X,Y,Z) coordinate system is fixed in space, with the Y-axis vertically upward, and the X, Z-plane in the undisturbed free surface of the water. The (x,y,z) coordinate system moves with the ship in such a way that the x, z-plane coincides with the X, Z-plane, and the y-axis is vertically upward and passes through the center of gravity of the ship. (See Figure 1.) The course of the ship is the motion of the origin of the moving (x,y,z) coordinate system. The x-axis is always tangent to the course of the ship.

The ship is regarded as a rigid body, and a third coordinate system (x',y',z') is fixed in the ship, with the x', y'-plane in the fore-and-aft plane of symmetry of the ship's hull, and the y'-axis passing through the

* Numbers in brackets refer to the bibliography.

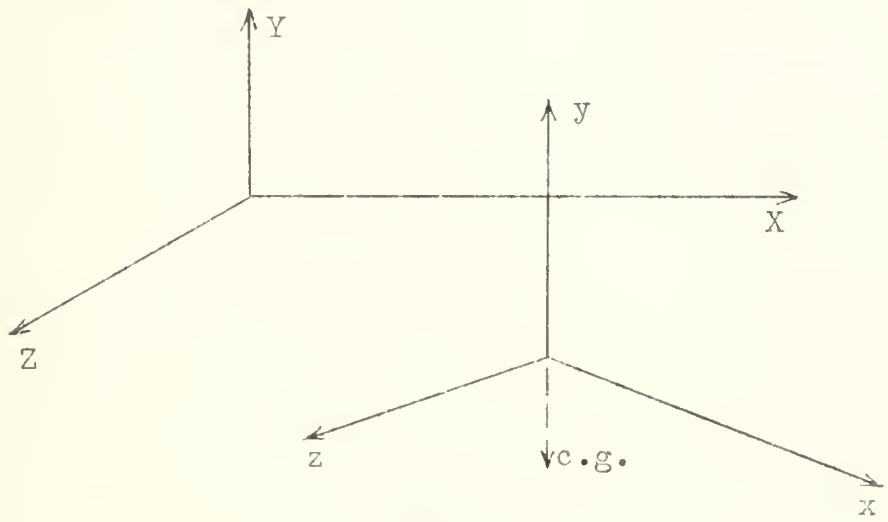

Figure 1.

center of gravity of the ship. (See Figure 2.) The (x',y',z') system coincides with the (x,y,z) system when the ship and water are at rest in their equilibrium positions.

The perturbation system is begun by writing the equations of the ship's hull relative to the coordinate system fixed in the ship in the form

$$z' = \beta h(x',y') \ ,$$

where β is a small dimensionless parameter (the breadth-length ratio of the ship). All quantities dealt with are transformed into the (x,y,z) coordinate system, and developed relative to the parameter β . The motion of the water is assumed to be given by a velocity potential $\phi(x,y,z,t;\beta)$, which, in turn, is assumed to possess the

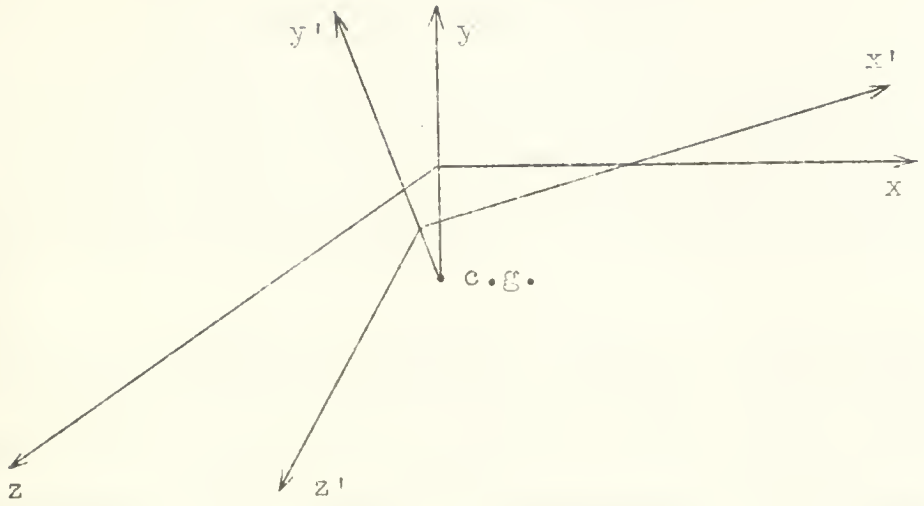

Figure 2.

development

$$\phi(x,y,z,t;\beta) = \beta\phi_1(x,y,z,t) + \beta^2\phi_2(x,y,z,t) + \ldots \quad .$$

The following problem then arises for the function $\phi_1(x,y,z,t)$, the first order term in the development of the velocity potential:

$$\phi_{1xx} + \phi_{1yy} + \phi_{1zz} = 0 \tag{1.1}$$

in the domain D - A, where D is the half space $y < 0$, and A is the plane disk which is the projection of the hull of the ship in its equilibrium position into the x, y-plane.

$$\phi_{1z} = s_o h_x(x,y) + F(x,y,t) \tag{1.2}$$

on A+,

(1.3) $$\phi_{1z} = - s_o h_x(x,y) + F(x,y,t)$$

on A-; where A+, A- are the two sides z = 0+, z = 0- of the plane disk A. The constant s_o is the zero order term in the development of the speed of the ship, and

$$F(x,y,t) = - s_o \theta_{21}(t) + x[\omega_1(t) + \dot{\theta}_{21}(t)] - \dot{\theta}_{11}(t)[y - y'_c],$$

where $\theta_{11}(t)$, $\theta_{21}(t)$ are the first order terms in the development of the angular displacements of the ship relative to the x- and y-axes, respectively; $\omega_1(t)$ is the first order term in the development of the angular speed of the ship, and the constant y'_c is the coordinate of the center of gravity in the (x',y',z') system. Dots over symbols denote differentiation with respect to t.

The free surface condition is

(1.4) $$s_o^2 \phi_{1xx} - 2s_o \phi_{1xt} + g \phi_{1y} + \phi_{1tt} = 0$$

on y = 0, where g is the acceleration of gravity.

If the ship moves in a seaway that consists of a single train of simple harmonic progressing waves, with circular frequency σ, which have their crests at right angles to the course of the ship, it turns out that the function F(x,y,t) is identically zero, and the problem is solved for this case in the above-mentioned paper[10] for a thin ship of otherwise arbitrary shape.

The case considered in this paper is that in which the wave crests quarter the ship at an arbitrary angle

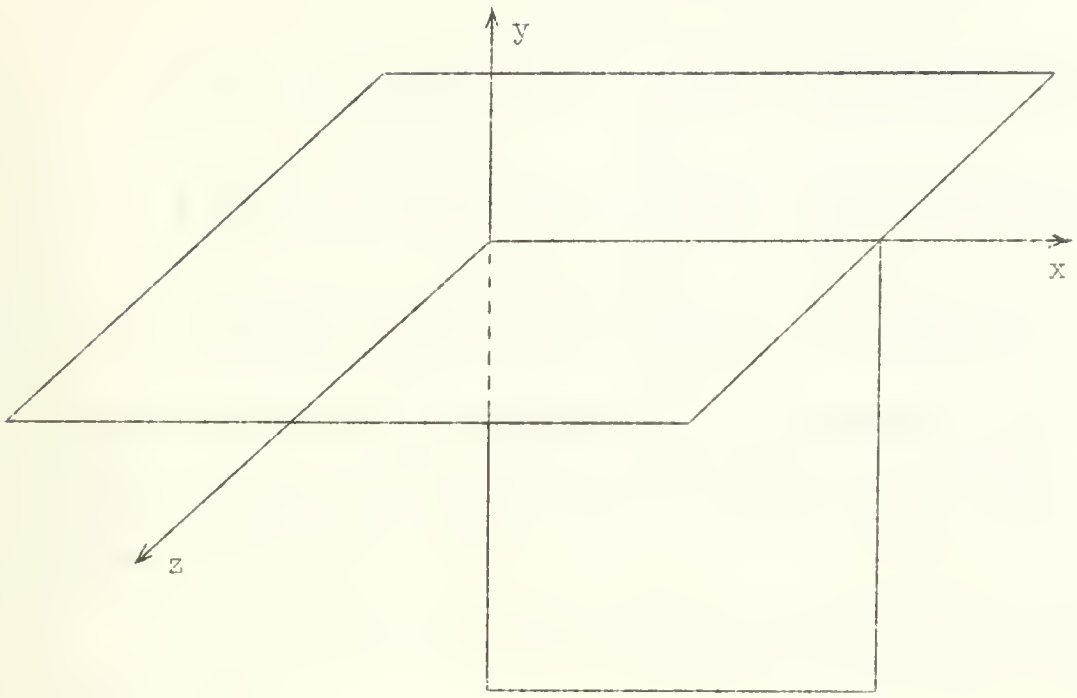

Figure 3.

a, and where the disk A comprises the quarter plane $z = 0$, $x > 0$, $y < 0$. (See Figure 3.)

The problem, as originally formulated, was first to find the function $\phi_1(x,y,z,t)$ from the partial differential equation (1.1) and the boundary conditions (1.2), (1.3) and (1.4). The function ϕ_1 would then be expressed in terms of $\theta_{11}(t)$, $\theta_{21}(t)$ and $\omega_1(t)$ as parameters,

$$\phi_1 = \phi_1[x,y,z,t; \theta_{11}(t); \theta_{21}(t); \omega_1(t)] .$$

These parameters were then to be determined from a system of ordinary differential equations

$$\int_{A+} (\phi_{1t} - s_o\phi_{1x})dx\,dy - \int_{A-} (\phi_{1t} - s_o\phi_{1x})dx\,dy = 0$$

$$\int_{A+} x(\phi_{1t} - s_o\phi_{1x})dx\,dy - \int_{A-} x(\phi_{1t} - s_o\phi_{1x})dx\,dy = 0$$

$$\int_{A+} y(\phi_{1t} - s_c\phi_{1x})dx\,dy - \int_{A-} y(\phi_{1t} - s_o\phi_{1x})dx\,dy = 0\ ,$$

which arises from the principle of the motion of the center of gravity, and the law of conservation of angular momentum.

The problem is simplified here by taking $F(x,y,t)$ to be known.

The function $\phi_1(x,y,z,t)$ is written as the sum of two harmonic functions

$$\phi_1(x,y,z,t) = P(x,y,z) + Q(x,y,z,t),$$

where

$$P_{xx} + P_{yy} + P_{zz} = 0 \text{ in } D - A,$$
$$s_o^2 P_{xx} + g\,P_y = 0 \quad \text{on the plane } y = 0,$$
$$P_z = s_o h_x(x,y) \quad \text{on } A+,$$
$$P_z = -\,s_o h_x(x,y) \quad \text{on } A-,$$

so that

(1.5) $\quad Q_{xx} + Q_{yy} + Q_{zz} = 0 \qquad$ in $D - A$,

(1.6) $\quad s_o^2 Q_{xx} - 2s_o Q_{xt} + g\,Q_y + Q_{tt} = 0 \qquad$ on $y = 0$,

(1.7) $\quad Q_z = F(x,y,t) \qquad$ on $A+$ and on $A-$.

Imposed on Q is the condition that Q behave at infinity like a simple harmonic plane wave coming in

at an angle a to the course of the ship

$$(1.8)\qquad Q \sim \operatorname{Re} \exp\left\{\frac{\sigma^2}{g}[y + i(x\cos a + z\sin a)] + i\left(\sigma' + \frac{s_o\sigma^2}{g}\cos a\right)t\right\}$$

for large $x^2 + z^2$.

The problem for $P(x,y,z)$ is solved in the paper[10] by Stoker and Peters. In this paper the problem for $Q(x,y,z,t)$ will be solved.

The function $Q(x,y,z,t)$ is assumed to be simple harmonic in time

$$Q(x,y,z,t) = \operatorname{Re}\, e^{i\Omega t}X(x,y,z)$$

where

$$\Omega = \sigma' + \frac{s_o\sigma^2}{g}\cos a.$$

Similarly

$$F(x,y,t) = \operatorname{Re}\, e^{i\Omega t}f(x,y)$$

From equations (1.5), (1.6), (1.7) and (1.8), it therefore follows that

$$(1.9)\qquad X_{xx} + X_{yy} + X_{zz} = 0 \qquad \text{in } D - A.$$

$$(1.10)\qquad s_o^2X_{xx} - 2is_o\Omega X_x + g\,X_y - \Omega^2X = 0 \qquad \text{on } y = 0.$$

$$(1.11)\qquad X_z = f(x,y) \qquad \text{on } A+ \text{ and on } A-.$$

$$(1.12)\qquad X \sim \exp\left\{\frac{\sigma^2}{g}[y + i(x\cos a + z\sin a)]\right\}$$

for large $x^2 + z^2$.

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N.Y.

2. Homogeneous Case

The case where $f(x,y)$ is identically zero is considered first. Setting

$$s_o^2 X_{xx} - 2is_o\Omega X_x + g\,X_y - \Omega^2 X = X^*(x,y,z)$$

for $y \leq 0$, results in the following boundary value problem [from (1.9), (1.10), (1.11) and (1.12)]:

$$X^*_{xx} + X^*_{yy} + X^*_{zz} = 0 \quad \text{in } D - A.$$
$$X^* = 0 \quad \text{on } y = 0.$$
$$X^*_z = 0 \quad \text{on } A+ \text{ and on } A-.$$
$$X^* \sim 0 \quad \text{for large } x^2 + z^2 .$$

This suggests that $X^*(x,y,z)$ be taken to be identically zero for $y \leq 0$. Doing so, and introducing cylindrical coordinates

$$x = r\cos\theta, \quad z = r\sin\theta, \quad y = y ,$$

and using the same symbol X to denote the function $X(x,y,z)$ and the function when expressed in cylindrical coordinates, $X(r,\theta,y)$, yields the following problem for $X(r,\theta,y)$:

From the fact that $X^*(x,y,z) = 0$ for $y \leq 0$,

$$L[X] = 0 \tag{2.1}$$

for $y \leq 0$, where L is the differential operator

$$L = s_o^2\left[\cos^2\theta\,\frac{\partial^2}{\partial r^2} - \frac{\sin 2\theta}{r}\,\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\,\frac{\partial^2}{\partial\theta^2} + \frac{\sin^2\theta}{r}\,\frac{\partial}{\partial r} + \frac{\sin 2\theta}{r^2}\,\frac{\partial}{\partial\theta}\right] - 2is_o\Omega\left[\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\right] + g\,\frac{\partial}{\partial y} - \Omega^2 \tag{2.2}$$

From (1.9)

$$X_{rr} + \frac{1}{r} X_r + \frac{1}{r^2} X_{\theta\theta} + X_{yy} = 0 \tag{2.3}$$

for $y < 0$, except on $\theta = 0$ and on $\theta = 2\pi$.

From (1.11)

$$X_\theta = 0 \tag{2.4}$$

on $\theta = 0$ and on $\theta = 2\pi$.

From (1.12),

$$X \sim \exp\left\{ \frac{\sigma^2}{g} [y + ir \cos (\theta - a)] \right\} \tag{2.5}$$

for large r.

From (2.4), it follows that $X(r,\theta,y)$ can be continued as a function of period 4π in θ. The finite Fourier transform $\bar{X}$ of X is introduced by the formula

$$\bar{X}(r,y,n) = \frac{1}{2\pi} \int_0^{2\pi} X(r,\theta,y) \cos \frac{n\theta}{2} \, d\theta \, .$$

Since $X_\theta = 0$ on $\theta = 0$ and on $\theta = 2\pi$, it follows that the transform of $X_{\theta\theta}$ is given by

$$\bar{X}_{\theta\theta} = - \frac{n^2}{4} \bar{X} \, .$$

Transforming (2.3) therefore yields

$$\bar{X}_{rr} + \frac{1}{r} \bar{X}_r - \frac{n^2}{4r^2} \bar{X} + \bar{X}_{yy} = 0$$

for $y < 0$.

The technique of separation of variables is used to find $\bar{X}(r,y,n)$. Setting

$$\bar{X}(r,y,n) = M(r,n)N(y)$$

and substituting in (3.3) yields

$$\frac{M_{rr} + \frac{1}{r} M_r - \frac{n^2}{4r^2} M}{M} = - \frac{N_{yy}}{N} = - k^2 .$$

Taking $k > 0$, and requiring that X go to zero as y goes to $- \infty$, it follows that N(y) is a constant multiple of e^{ky}. The function M(r,n) solves the differential equation

$$M_{rr} + \frac{1}{r} M_r + (k^2 - \frac{n^2}{4r^2})M = 0,$$

which is Bessel's equation of order $\frac{n}{2}$ for the argument kr. If the function X is required to be bounded for $r = 0$, the only solution of this differential equation is

$$M(r,n) = b_n J_{\frac{n}{2}}(kr),$$

where b_n is a constant. The asymptotic expression for $X(r,\theta,y)$ suggests that $k = \frac{\sigma^2}{g}$, so that

$$\bar{X} = b_n \, e^{ky} \, J_{\frac{n}{2}}(kr).$$

Inverting the finite Fourier transform yields

$$X(r,\theta,y) = e^{ky} \sum_{n=0}^{\infty} \varepsilon_n b_n \, J_{\frac{n}{2}}(kr) \cos \frac{n\theta}{2} \tag{2.6}$$

where

$$\varepsilon_n = \begin{cases} 1 & \text{for } n = 0 \\ 2 & \text{for } n > 0 , \end{cases}$$

and the constants b_n are to be determined using (2.1).

Applying the operator L to (2.6) yields

$$L[X] = e^{ky} \sum_{n=0}^{\infty} \varepsilon_n b_n \Bigg\{ s_o^2 [k^2 J''_{\frac{n}{2}}(kr) \cos^2\theta \cos\frac{n\theta}{2}$$

$$+ \frac{nk}{2r} J'_{\frac{n}{2}}(kr) \sin 2\theta \sin\frac{n\theta}{2}$$

$$- \frac{n^2}{4r^2} J_{\frac{n}{2}}(kr) \sin^2\theta \cos\frac{n\theta}{2}$$

$$+ \frac{k}{r} J'_{\frac{n}{2}}(kr) \sin^2\theta \cos\frac{n\theta}{2}$$

$$- \frac{n}{2r^2} J_{\frac{n}{2}}(kr) \sin 2\theta \sin\frac{n\theta}{2}]$$

$$- 2is_o\Omega[kJ'_{\frac{n}{2}}(kr) \cos\theta \cos\frac{n\theta}{2}$$

$$+ \frac{n}{2r} J_{\frac{n}{2}}(kr) \sin\theta \sin\frac{n\theta}{2}]$$

$$+ (\sigma^2 - \Omega^2) J_{\frac{n}{2}}(kr) \cos\frac{n\theta}{2} \Bigg\} .$$

Using the identities

$$\cos^2\theta \cos\frac{n\theta}{2} = \frac{1}{2}\cos\frac{n\theta}{2} + \frac{1}{4}\cos\frac{(n-4)\theta}{2} + \frac{1}{4}\cos\frac{(n+4)\theta}{2}$$

$$\sin 2\theta \sin\frac{n\theta}{2} = \frac{1}{2}\cos\frac{(n-4)\theta}{2} - \frac{1}{2}\cos\frac{(n+4)\theta}{2}$$

$$\sin^2\theta \cos\frac{n\theta}{2} = \frac{1}{2}\cos\frac{n\theta}{2} - \frac{1}{4}\cos\frac{(n-4)\theta}{2} - \frac{1}{4}\cos\frac{(n+4)\theta}{2}$$

$$\cos\theta \cos\frac{n\theta}{2} = \frac{1}{2}\cos\frac{(n-2)\theta}{2} + \frac{1}{2}\cos\frac{(n+2)\theta}{2}$$

$$\sin\theta \sin\frac{n\theta}{2} = \frac{1}{2}\cos\frac{(n-2)\theta}{2} - \frac{1}{2}\cos\frac{(n+2)\theta}{2}$$

$$L[X] = e^{ky} \sum_{n=0}^{\infty} \varepsilon_n b_n \Big\{ s_o^2 k^2 J_{\frac{n}{2}}''(kr)\left[\tfrac{1}{2}\cos\tfrac{n\theta}{2} + \tfrac{1}{4}\cos\tfrac{(n-4)\theta}{2} + \tfrac{1}{4}\cos\tfrac{(n+4)\theta}{2}\right] + s_o^2\left[\tfrac{nk}{2r} J_{\frac{n}{2}}'(kr) - \tfrac{n}{2r^2} J_{\frac{n}{2}}(kr)\right]\cdot$$

$$\left[\tfrac{1}{2}\cos\tfrac{(n-4)\theta}{2} - \tfrac{1}{2}\cos\tfrac{(n+4)\theta}{2}\right]$$

$$+ s_o^2\left[\tfrac{k}{r} J_{\frac{n}{2}}'(kr) - \tfrac{n^2}{4r^2} J_{\frac{n}{2}}(kr)\right]\left[\tfrac{1}{2}\cos\tfrac{n\theta}{2} - \tfrac{1}{4}\cos\tfrac{(n-4)\theta}{2} - \tfrac{1}{4}\cos\tfrac{(n+4)\theta}{2}\right]$$

$$-2is_o k\Omega J_{\frac{n}{2}}'(kr)\left[\tfrac{1}{2}\cos\tfrac{(n-2)\theta}{2} + \tfrac{1}{2}\cos\tfrac{(n+2)\theta}{2}\right] - \frac{is_o n\Omega}{r} J_{\frac{n}{2}}(kr)\left[\tfrac{1}{2}\cos\tfrac{(n-2)\theta}{2} - \tfrac{1}{2}\cos\tfrac{(n+2)\theta}{2}\right]$$

$$+ (\sigma^2 - \Omega^2) J_{\frac{n}{2}}(kr)\cos\tfrac{n\theta}{2} \Big\}$$

$$= e^{ky} \sum_{n=0}^{\infty} \varepsilon_n b_n \Big\{ \frac{s_o^2}{2r^2}\cos\tfrac{n\theta}{2}\left[k^2r^2 J_{\frac{n}{2}}''(kr) + kr\, J_{\frac{n}{2}}'(kr) - \tfrac{n^2}{4} J_{\frac{n}{2}}(kr)\right]$$

$$+ \frac{s_o^2}{4r^2}\cos\tfrac{(n-4)\theta}{2}\left[k^2r^2 J_{\frac{n}{2}}(kr) + kr(n-1) J_{\frac{n}{2}}'(kr) + \tfrac{n(n-4)}{4} J_{\frac{n}{2}}(kr)\right] + \frac{s_o^2}{4r^2}\cos\tfrac{(n+4)\theta}{2}\left[k^2r^2 J_{\frac{n}{2}}''(kr) - kr(n+1) J_{\frac{n}{2}}'(kr) + \tfrac{n(n+4)}{4} J_{\frac{n}{2}}(kr)\right]$$

$$- \frac{is_o\Omega}{r}\cos\tfrac{(n-2)\theta}{2}\left[kr\, J_{\frac{n}{2}}'(kr) + \tfrac{n}{2} J_{\frac{n}{2}}(kr)\right] - \frac{is_o\Omega}{r}\cos\tfrac{(n+2)\theta}{2}\left[kr\, J_{\frac{n}{2}}'(kr) - \tfrac{n}{2} J_{\frac{n}{2}}(kr)\right]$$

$$+ (\sigma^2 - \Omega^2) J_{\frac{n}{2}}(kr)\cos\tfrac{n\theta}{2} \Big\}$$

Substituting the identities

$$k^2r^2J''_{\frac{n}{2}}(kr) + kr\,J'_{\frac{n}{2}}(kr) + (k^2r^2 - \tfrac{n^2}{4})\,J_{\frac{n}{2}}(kr) = 0$$

$$kr\,J'_{\frac{n}{2}}(kr) + \tfrac{n}{2}\,J_{\frac{n}{2}}(kr) = kr\,J_{\frac{n-2}{2}}(kr)$$

$$kr\,J'_{\frac{n}{2}}(kr) - \tfrac{n}{2}\,J_{\frac{n}{2}}(kr) = -\,kr\,J_{\frac{n+2}{2}}(kr)$$

$$\frac{n-2}{kr}\,J_{\frac{n-2}{2}}(kr) - J_{\frac{n}{2}}(kr) = J_{\frac{n-4}{2}}(kr)$$

$$\frac{n+2}{kr}\,J_{\frac{n+2}{2}}(kr) - J_{\frac{n}{2}}(kr) = J_{\frac{n+4}{2}}(kr)\ ,$$

and collecting terms, yields

$$\begin{aligned}(2.7)\quad L[X] = e^{ky}\sum_{n=0}^{\infty}\varepsilon_n b_n \Bigg\{ &\frac{s_o^2k^2}{4}\Big[J_{\frac{n-4}{2}}(kr)\cos\frac{(n-4)\theta}{2} \\ &- 2\,J_{\frac{n}{2}}(kr)\cos\frac{n\theta}{2} + J_{\frac{n+4}{2}}(kr)\cos\frac{(n+4)\theta}{2}\Big] \\ &- is_ok\Omega\Big[J_{\frac{n-2}{2}}(kr)\cos\frac{(n-2)\theta}{2} - J_{\frac{n+2}{2}}(kr)\cos\frac{(n+2)\theta}{2}\Big] \\ &+ (\sigma^2 - \Omega^2)\,J_{\frac{n}{2}}(kr)\cos\frac{n\theta}{2}\Bigg\}\end{aligned}$$

Taking a finite Fourier transform of (2.1) [with L[X] given by (2.7)],

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} L[X]\cos\frac{m\theta}{2}\,d\theta = e^{ky}J_{\frac{m}{2}}(kr)\,[&s_o^2k^2(b_{m+4} - 2\,b_m + b_{m-4}) \\ &- is_ok\Omega(b_{m+2} - b_{m-2}) + (\sigma^2 - \Omega^2)b_m] = 0\ ,\end{aligned}$$

and replacing k by its value $\frac{\sigma^2}{g}$, there results the difference equation

$$(2.8)\quad \frac{s_o^2\sigma^4}{4g^2}(b_{m+4} - 2\,b_m + b_{m-4}) - \frac{is_o\Omega\sigma^2}{g}(b_{m+2} - b_{m-2}) + (\sigma^2 - \Omega^2)b_m = 0.$$

Substituting $b_m = e^{ipm}$ in (2.8), yields

$$\frac{s_o^2\sigma^4}{4g^2}(e^{4ip} - 2 + e^{-4ip}) - \frac{is_o\Omega\sigma^2}{g}(e^{2ip} - e^{-2ip}) + (\sigma^2 - \Omega^2) = 0,$$

a quadratic equation in $e^{2ip} - e^{-2ip}$, a solution of which is

$$(2.9)\qquad e^{2ip} - e^{-2ip} = \frac{2ig}{s_o\sigma^2}(\Omega - \sigma).$$

Replacing Ω by its value $\sigma + \frac{s_o\sigma^2}{g}\cos a$ in (2.9),

$$e^{2ip} - e^{-2ip} = 2i\cos a$$

which, in turn, is a quadratic equation in e^{2ip}, whose solutions are

$$e^{2ip} = i\cos a \pm \sin a = i\,e^{\mp ia} = \exp(\tfrac{i\pi}{2} \pm ia),$$

so that $p = \frac{\pi}{4} \pm \frac{a}{2}$. In particular, selecting

$$b_m = \exp(\tfrac{im\pi}{4})\cos\tfrac{ma}{2},$$

$$(2.10)\quad X(r,\theta,y) = \exp(\tfrac{\sigma^2 y}{g})\sum_{n=0}^{\infty}\varepsilon_n \exp(\tfrac{in\pi}{4})J_{\frac{n}{2}}(\tfrac{\sigma^2 r}{g})\cdot\cos\tfrac{na}{2}\cos\tfrac{n\theta}{2}.$$

That the function $X(r,\theta,y)$ given in (2.10) satisfies (2.1), (2.3) and (2.4), can be verified without difficulty.

That $X(r,\theta,y)$ has the proper asymptotic character (2.5), can be seen following the method used by Peters and Stoker in a paper[9] on diffraction problems. There it is shown that

$$\sum_{n=0}^{\infty} \varepsilon_n \exp\left(\frac{in\pi}{4}\right) J_{\frac{n}{2}}(r) \cos\frac{n\alpha}{2} \cos\frac{n\theta}{2} \sim \exp\left[ir\cos(\theta - \alpha)\right]$$

for large r.

A rather weak uniqueness theorem can be stated, namely that the solution $X(r,\theta,y)$ is unique in the class of functions of the form

$$X(r,\theta,y) = e^{my} S(r,\theta),$$

where m is a constant.

If it is assumed that $X(r,\theta,y)$ is of this form, it follows that $m = \frac{\sigma^2}{g}$, from (2.5); so that $S(r,\theta)$ solves what is essentially Sommerfeld's diffraction problem. That this problem has a unique solution is shown in the same paper[9] under the one additional assumption

$$S(r,\theta) \sim \frac{c}{r^{1+b}}$$

near $r = 0$; where b and c are constants, with $0 < b < 1$.

3. Inhomogeneous Case

The case where $f(x,y)$ is not identically zero is now considered. Introducing cylindrical coordinates as before, and defining $X(r,\theta,y)$ exactly as in the homogeneous case, equations (1.9) and (1.10) become respectively

$$(3.1) \qquad X_{rr} + \frac{1}{r} X_r + \frac{1}{r^2} X_{\theta\theta} + X_{yy} = 0 \text{ in } D - A,$$

$$(3.2) \qquad L[X] = 0 \text{ on } y = 0,$$

where the operator L is given in (2.2).

Letting $f(r,y)$ denote the function $f(r \cos \theta, y)$, evaluated on $\theta = 0$ (or on $\theta = 2\pi$), equation (1.11) becomes

$$(3.3) \qquad X_\theta = r\, f(r,y) \text{ on } \theta = 0 \text{ and on } \theta = 2\pi.$$

The function $X(r,\theta,y)$ is written as the sum of two harmonic functions

$$(3.4) \qquad X(r,\theta,y) = u(r,\theta,y) + v(r,\theta,y),$$

where $u(r,\theta,y)$ solves the problem:

$$(3.5) \qquad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{yy} = 0$$

for $y < 0$, except on $\theta = 0$ and on $\theta = 2\pi$.

$$(3.6) \qquad u = 0 \text{ on } y = 0.$$

$$(3.7) \qquad u_\theta = r\, f(r,y) \text{ on } \theta = 0 \text{ and on } \theta = 2\pi.$$

Then $v(r, \theta, y)$ solves the problem:

$$(3.8) \qquad v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} + v_{yy} = 0$$

for $y < 0$, except on $\theta = 0$ and on $\theta = 2\pi$.

$$(3.9) \qquad L[v] = - g\, u_y(r,\theta,0) \text{ on } y = 0,$$

since $L[u] = g\, u_y$, from the form of the operator L in rectangular coordinates.

$$(3.10) \qquad v_\theta = 0 \text{ on } \theta = 0 \text{ and on } \theta = 2\pi.$$

Set

$$(3.11) \qquad L[v] = w(r,\theta,y)$$

in $y \leq 0$. Then

$$(3.12) \quad w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} + w_{yy} = 0$$

for $y < 0$, except on $\theta = 0$ and on $\theta = 2\pi$. (This, too, follows from the form of the operator L in rectangular coordinates.)

$$(3.13) \quad w = - g\, u_y(r,\theta,0)$$

on $y = 0$, from the definition of $w(r,\theta,y)$.

$$(3.14) \quad w_\theta = 0$$

on $\theta = 0$ and on $\theta = 2\pi$, since L is a tangential operator relative to the planes $\theta = 0$ and $\theta = 2\pi$; as is again evident from the form of the operator L in rectangular coordinates.

The procedure in finding $X(r,\theta,y)$ will be the following. First a function $u(r,\theta,y)$ will be found having properties (3.5), (3.6) and (3.7). With $u(r,\theta,y)$ known, a function $w(r,\theta,y)$ will then be found having properties (3.12), (3.13) and (3.14). With $w(r,\theta,y)$ known, a solution $v(r,\theta,y)$ of the differential equation (3.11) will be found which also satisfies (3.8), (3.9) and (3.10).

To this end, observe that the function $u(r,\theta,y)$ is harmonic for $y < 0$, with $u = 0$ on $y = 0$ [equations (3.5), (3.6)]. Therefore $u(r,\theta,y)$ may be continued harmonically as a function odd in y,

$$u(r,\theta,y) = - u(r,\theta, - y) \quad \text{for } y > 0 .$$

If $f(r,y)$ is defined for $y > 0$ by

$$f(r,y) = - f(r, - y) ,$$

then, from (3.5) and (3.7) it follows that $u(r,\theta,y)$ solves the problem

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{yy} = 0 , \tag{3.15}$$

except on $\theta = 0$ and on $\theta = 2\pi$, and

$$u_\theta = r\, f(r,y) \tag{3.16}$$

on $\theta = 0$ and on $\theta = 2\pi$.

To find $u(r,\theta,y)$, a Green's function $G(r_o,\theta_o,y_o;r,\theta,y)$, with the following properties, is constructed:

$$\frac{\partial^2 G}{\partial r_o^2} + \frac{1}{r_o}\frac{\partial G}{\partial r_o} + \frac{1}{r_o^2}\frac{\partial^2 G}{\partial \theta_o^2} + \frac{\partial^2 G}{\partial y_o^2} = -\frac{4\pi}{r_o}\delta(r_o - r)\delta(\theta_o - \theta)\delta(y_o - y), \tag{3.17}$$

except on $\theta_o = 0$ and on $\theta_o = 2\pi$; and

$$\frac{\partial G}{\partial \theta_o} = 0 \tag{3.18}$$

on $\theta_o = 0$ and on $\theta_o = 2\pi$.

Equation (3.18) implies that G can be continued harmonically as a function of period 4π in θ_o . Setting

$$\bar{G}(r_o,n,y_o;r,\theta,y) = \frac{1}{2\pi}\int_0^{2\pi} G(r_o,\theta_o,y_o;r,\theta,y)\cos\frac{n\theta_o}{2}\, d\theta_o$$

and taking a finite Fourier transform of (3.17) yields

$$\frac{\partial^2 \bar{G}}{\partial r_o^2} + \frac{1}{r_o}\frac{\partial \bar{G}}{\partial r_o} - \frac{n^2}{4r_o^2}\bar{G} + \frac{\partial^2 \bar{G}}{\partial y_o^2} = -\frac{2}{r_o}\cos\frac{n\theta}{2}\,\delta(r_o - r)\delta(y_o - y). \tag{3.19}$$

Setting

$$\bar{\bar{G}}(r_o,n,q;r,\theta,y) = \int_{-\infty}^{\infty} \exp(iqy_o)\bar{G}(r_o,n,y_o;r,\theta,y)dy_o ,$$

and taking a Fourier transform of (3.19), yields

$$(3.20)\quad \frac{\partial^2\bar{\bar{G}}}{\partial r_o^2} + \frac{1}{r_o}\frac{\partial\bar{\bar{G}}}{\partial r_o} - \left(\frac{n^2}{4r_o^2} + q^2\right)\bar{\bar{G}}$$

$$= -\frac{2}{r_o}\exp(iqy)\cos\frac{n\theta}{2}\,\delta(r_o - r)$$

Setting

$$\bar{\bar{\bar{G}}}(p,n,q;r,\theta,y) = \int_0^{\infty} \bar{\bar{G}}(r_o,n,q;r,\theta,y)J_{\frac{n}{2}}(r_o p)r_o dr_o ,$$

taking a Hankel transform of (3.20), and collecting terms, yields

$$\bar{\bar{\bar{G}}}(p,n,q;r,\theta,y) = \frac{2\exp(iqy)\cos\frac{n\theta}{2}\,J_{\frac{n}{2}}(rp)}{p^2 + q^2}$$

Inverting the Hankel transform

$$\bar{\bar{G}}(r_o,n,q;r,\theta,y) = 2\exp(iqy)\cos\frac{n\theta}{2}\int_0^{\infty}\frac{J_{\frac{n}{2}}(rp)J_{\frac{n}{2}}(r_o p)p\,dp}{p^2 + q^2}$$

Inverting the Fourier transform

$$\bar{G}(r_o,n,y_o;r,\theta,y) =$$

$$\frac{1}{\pi}\cos\frac{n\theta}{2}\int_{-\infty}^{\infty}\exp[iq(y-y_o)]\int_0^{\infty}\frac{J_{\frac{n}{2}}(rp)J_{\frac{n}{2}}(r_o p)p\,dp\,dq}{p^2 + q^2} .$$

Inverting the finite Fourier transform,

$$(3.21)\quad G(r_o,\theta_o,y_o;r,\theta,y) = \frac{1}{\pi}\sum_{n=0}^{\infty} \epsilon_n \cos\frac{n\theta}{2}\cos\frac{n\theta_o}{2} \cdot$$

$$\int_{-\infty}^{\infty} \exp\,[iq(y-y_o)] \int_0^{\infty} \frac{J_{\frac{n}{2}}(rp)J_{\frac{n}{2}}(r_o p)p\,dp\,dq}{p^2+q^2} \cdot$$

The function $u(r,\theta,y)$ is now found by applying Green's second identity to the functions $u(r,\theta,y)$ and $G(r_o,\theta_o,y_o;r,\theta,y)$ in the domain $y < 0$, $0 < \theta < 2\pi$. Using the oddness of the function $f(r,y)$

$$(3.22)\quad u(r,0,y) = -\frac{1}{4\pi}\int_{-\infty}^{0}\int_0^{\infty} [G^*(r_o,y_o;r,\theta,y)$$

$$- G^*(r_o,-y_o;r,\theta,y)]\, f(r_o,y_o)dr_o\,dy_o \,,$$

where

$$G^*(r_o,y_o;r,\theta,y) = G(r_o,0,y_o;r,\theta,y) - G(r_o,2\pi,y_o;r,\theta,y)$$

Equation (3.13) can therefore be written

$$(3.23)\quad w(r,\theta,0) = \frac{g}{4\pi}\int_{-\infty}^{0}\int_0^{\infty} f(r_o,y_o)\frac{\partial}{\partial y}[G^*(r_o,y_o;r,\theta,y)$$

$$- G^*(r_o,-y_o;r,\theta,y)]\Big|_{y=0} dr_o\,dy_o \,.$$

From (3.21), it follows that

$$G^*(r_o,-y_o;r,\theta,y) = G^*(r_o,y_o;r,\theta,-y), \text{ and}$$

$$\frac{\partial}{\partial y}[G^*(r_o,y_o;r,\theta,y)] = -\frac{\partial}{\partial y}[G^*(r_o,y_o;r,\theta,-y)] \text{ on } y = 0.$$

Substituting in (3.23),

$$(3.24)\quad w(r,\theta,0) = -\frac{g}{2\pi}\int_{-\infty}^{0}\int_{0}^{\infty} f(r_o,y_o)\,\frac{\partial}{\partial y}\,[G^*(r_o,y_o;\theta,-y)]\Big|_{y=0}\,dr_o\,dy_o\,.$$

The form of (3.24) suggests that the function

$$(3.25)\quad w(r,\theta,y) = -\frac{g}{2\pi}\int_{-\infty}^{0}\int_{0}^{\infty} f(r_o,y_o)\,\frac{\partial}{\partial y}\,[G^*(r_o,y_o;r,\theta,-y)]\,dr_o\,dy_o$$

be examined. It is seen without difficulty that this function has properties (3.12), (3.13) and (3.14). Using (3.21) and the definition of $G^*(r_o,y_o;r,\theta,y)$, equation (3.25) can be written

$$w(r,\theta,y) = -\frac{g}{2\pi}\int_{-\infty}^{0}\int_{0}^{\infty} f(r_o,y_o)\,\frac{\partial}{\partial y}\left\{\frac{1}{\pi}\sum_{n=0}^{\infty}\varepsilon_n\cos\frac{n\theta}{2}\,[1-(-)^n]\cdot\right.$$

$$\left.\int_{-\infty}^{\infty}\exp\,[-iq(y+y_o)]\int_{0}^{\infty}\frac{J_{\frac{n}{2}}(rp)\,J_{\frac{n}{2}}(r_op)\,p\,dp\,dq}{p^2+q^2}\right\}dr_o\,dy_o\,.$$

Since $p \geq 0$, $y < 0$ and $y_o \leq 0$, it follows from the theory of residues that

$$\int_{-\infty}^{\infty}\frac{\exp\,[-iq(y+y_o)]}{p^2+q^2}\,dq = \frac{\pi}{p}\exp\,[p(y+y_o)]$$

so that

$$(3.26)\quad w(r,\theta,y) = -\frac{g}{\pi}\int_{-\infty}^{0}\int_{0}^{\infty} f(r_o,y_o)\sum_{n=1}^{\infty}[1-(-)^n]\cos\frac{n\theta}{2}\cdot$$

$$\int_{0}^{\infty} e^{p(y+y_o)}\,J_{\frac{n}{2}}(rp)\,J_{\frac{n}{2}}(r_op)\,p\,dp\,dr_o\,dy_o\,.$$

The next step is to find a solution $v(r,\theta,y)$ of (3.11) which has properties (3.8), (3.9) and (3.10). Assuming that $v(r,\theta,y)$ has the same form as $w(r,\theta,y)$,

$$(3.27)\quad v(r,\theta,y) = -\frac{g}{\pi}\int_{-\infty}^{0}\int_{0}^{\infty} f(r_o,y_o)\sum_{n=1}^{\infty}[1-(-)^n]\cos\frac{n\theta}{2}\cdot \int_{0}^{\infty} e^{p(y+y_o)} J_{\frac{n}{2}}(rp)A_n(r_o,p)p\,dp\,dr_o\,dy_o\,,$$

where the functions $A_n(r_o,p)$ are to be determined from the relation (3.11).

Applying the operator L to $v(r,\theta,y)$, as given by (3.27), and substituting the trigonometric and Bessel identities, as in the homogeneous case, yields

$$(3.28)\quad L[v] = -\frac{g}{\pi}\int_{-\infty}^{0}\int_{0}^{\infty} f(r_o,y_o)\sum_{n=1}^{\infty}[1-(-)^n]\int_{0}^{\infty} e^{p(y+y_o)}A_n(r_o,p)\cdot$$
$$\left\{\frac{s_o^2p^2}{4}\left[\cos\frac{(n-4)\theta}{2}\,J_{\frac{n-4}{2}}(rp) - 2J_{\frac{n}{2}}(rp)\cos\frac{n\theta}{2}\right.\right.$$
$$\left. + \cos\frac{(n+4)\theta}{2}\,J_{\frac{n+4}{2}}(rp)\right] - is_op\Omega\left[J_{\frac{n-2}{2}}(rp)\cos\frac{(n-2)\theta}{2}\right.$$
$$\left.\left. - J_{\frac{n+2}{2}}(rp)\cos\frac{(n+2)\theta}{2}\right] + (gp-\Omega^2)J_{\frac{n}{2}}(rp)\cos\frac{n\theta}{2}\right\}$$
$$p\,dp\,dr_o\,dy_o\,.$$

Since $L[v] = w$, take finite transforms of $L[v]$ as given in (3.28), and of w as given in (3.26), and obtain

$$(3.29)\quad \int_{-\infty}^{0}\int_{0}^{\infty} f(r_o,y_o)[1-(-)^m]\int_{0}^{\infty} e^{p(y+y_o)} J_{\frac{m}{2}}(rp)\cdot$$

$$\left\{\frac{s_o^2p^2}{4}\,[A_{m+4}(r_o,p) - 2\,A_m(r_o,p) + A_{m-4}(r_o,p)]\right.$$

$$- is_op\Omega[A_{m+2}(r_o,p) - A_{m-2}(r_o,p)]$$

$$\left. + (gp - \Omega^2)A_m(r_o,p)\right\}\, p\,dp\,dr_o\,dy_o$$

$$= \int_{-\infty}^{0}\int_{0}^{\infty} f(r_o,y_o)\,[1-(-)^m]\cdot$$

$$\int_{0}^{\infty} e^{p(y+y_o)} J_{\frac{m}{2}}(rp)J_{\frac{m}{2}}(r_op)p\,dp\,dr_o\,dy_o\ .$$

A comparison of the left and right hand sides of (3.29) yields the inhomogeneous difference equation

$$(3.30)\quad \frac{s_o^2p^2}{4}\,[A_{m+4}(r_o,p) - 2\,A_m(r_o,p) + A_{m-4}(r_o,p)]$$

$$- is_op\Omega[A_{m+2}(r_o,p) - A_{m-2}(r_o,p)]$$

$$+ (gp - \Omega^2)A_m(r_o,p) = J_{\frac{m}{2}}(r_op).$$

The right hand side of (3.30) can be written

$$J_{\frac{m}{2}}(r_op) = \frac{\exp\left(\frac{im\pi}{2}\right)}{2\pi i}\int_P \exp\left[-\frac{pr_o}{2}\left(z - \frac{1}{z}\right)\right]z^{-1-m/2}\,dz$$

where the path P is

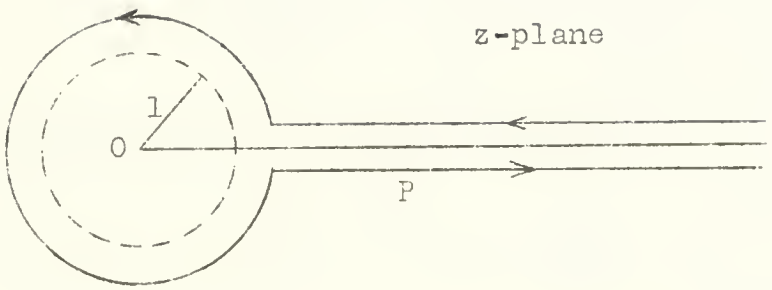

Figure 4.

Assuming that the solution of the difference equation is of the form

$$A_m(r_o,p) = \frac{\exp\left(\frac{im\pi}{2}\right)}{2\pi i} \int_P F(z,p) \exp\left[-\frac{pr_o}{2}\left(z - \frac{1}{z}\right)\right] z^{-1-m/2} dz ,$$

with $F(z,p)$ to be determined; and substituting in (3.30) yields

$$\frac{\exp\left(\frac{im\pi}{2}\right)}{2\pi i} \int_P F(z,p)\left\{\frac{s_o^2p^2}{4}\left(z - \frac{1}{z}\right) - is_op\Omega\left(z - \frac{1}{z}\right) + (gp - \Omega^2)\right\} .$$

$$\exp\left[-\frac{pr_o}{2}\left(z - \frac{1}{z}\right)\right] z^{-1-m/2} dz$$

$$= \frac{\exp\left(\frac{im\pi}{2}\right)}{2\pi i} \int_P \exp\left[-\frac{pr_o}{2}\left(z - \frac{1}{z}\right)\right] z^{-1-m/2} dz ,$$

so that

$$F(z,p) = \left[\frac{s_o^2p^2}{4}\left(z - \frac{1}{z}\right)^2 - is_op\Omega\left(z - \frac{1}{z}\right) + (gp - \Omega^2)\right]^{-1}$$

and

$$A_m(r_o,p) = \frac{\exp\left(\frac{im\pi}{2}\right)}{2\pi i} \int_P \frac{\exp\left[-\frac{pr_o}{2}(z-\frac{1}{z})\right] z^{-1-m/2} dz}{\frac{s_o^2 p^2}{4}(z-\frac{1}{z})^2 - is_o p\Omega(z-\frac{1}{z}) + (gp-\Omega^2)} .$$

Substituting this result in (3.27), yields

$$(3.31)\quad v(r,\theta,y) = -\frac{g}{\pi} \int_{-\infty}^{0} \int_0^{\infty} f(r_o,y_o) \sum_{n=1}^{\infty} [1-(-)^n] \cos\frac{n\theta}{2} \cdot \int_0^{\infty} \exp[p(y+y_o)] J_{\frac{n}{2}}(rp) \frac{\exp\left(\frac{in\pi}{2}\right)}{2\pi i} \cdot \int_P \frac{\exp\left[-\frac{pr_o}{2}(z-\frac{1}{z})\right] z^{-1-n/2} dz\, p\, dp\, dr_o\, dy_o}{\frac{s_o^2 p^2}{4}(z-\frac{1}{z})^2 - is_o p\Omega(z-\frac{1}{z}) + (gp-\Omega^2)} .$$

Replacing $J_{\frac{n}{2}}(rp)$ in (3.31) by

$$J_{\frac{n}{2}}(rp) = \frac{\exp\left(\frac{in\pi}{2}\right)}{2\pi i} \int_{P'} \exp\left[-\frac{pr}{2}(z'-\frac{1}{z'})\right](z')^{-1-n/2} dz' ,$$

where P' is the same path in the z'-plane as P is in the z-plane, yields

$$v(r,\theta,y) = -\frac{g}{\pi} \int_{-\infty}^{0} \int_0^{\infty} f(r_o,y_o) \sum_{n=1}^{\infty} [1-(-)^n] \cos\frac{n\theta}{2} \int_0^{\infty} \exp[p(y+y_o)] \cdot \frac{\exp\left(\frac{in\pi}{2}\right)}{2\pi i} \int_{P'} \exp\left[-\frac{pr}{2}(z'-\frac{1}{z'})\right](z')^{-1-n/2} dz' \cdot \frac{\exp\left(\frac{in\pi}{2}\right)}{2\pi i} \int_P \frac{\exp\left[-\frac{pr_o}{2}(z-\frac{1}{z})\right] z^{-1-n/2} dz\, p\, dp\, dr_o\, dy_o}{\frac{s_o^2 p^2}{4}(z-\frac{1}{z})^2 - is_o p\Omega(z-\frac{1}{z}) + (gp-\Omega^2)} ,$$

Summing over n,

(3.32)

$$v(r,\theta,y) = \frac{g\cos\frac{\theta}{2}}{2\pi^3}\int_{-\infty}^{0}\int_{0}^{\infty}\int_{0}^{\infty}\int_{P'}\int_{P}\frac{f(r_o,y_o)(zz')^{-1/2}(1-zz')}{(zz')^2 - 2zz'\cos\theta + 1}\cdot$$

$$\frac{\exp\left[p(y+y_o) - \frac{rp}{2}(z' - \frac{1}{z'}) - \frac{r_o p}{2}(z - \frac{1}{z})\right]}{\frac{s_o^2 p^2}{4}(z - \frac{1}{z})^2 - is_o p\Omega(z - \frac{1}{z}) + (gp - \Omega^2)}\cdot$$

$$p\,dz'\,dz\,dp\,dr_o\,dy_o\ .$$

The function $v(r,\theta,y)$ is thus constructed solving equation (3.11). That $v(r,\theta,y)$ also satisfies (3.8), (3.9) and (3.10) can be seen most easily using representation (3.31) for $v(r,\theta,y)$.

4. Summary

The mathematical problem considered is to find a function $X(r,\theta,y)$ with the following properties:

(4.1) $$X_{rr} + \frac{1}{r}X_r + \frac{1}{r^2}X_{\theta\theta} + X_{yy} = 0$$

for $y < 0$, except on $\theta = 0$ and on $\theta = 2\pi$.

(4.2) $$L[X] = \left\{ s_o^2\left[\cos^2\theta\,\frac{\partial^2}{\partial r^2} - \frac{\sin 2\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin 2\theta}{r^2}\frac{\partial}{\partial\theta}\right] - 2is_o\Omega\left[\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right] + g\frac{\partial}{\partial y} - \Omega^2 \right\} X(r,\theta,y) = 0$$

on $y = 0$.

(4.3) $\qquad X_\theta = r\, f(r,y)$

on $\theta = 0$ and on $\theta = 2\pi$.

(4.4) $\qquad X(r,\theta,y) \sim \exp\left\{ \frac{\sigma^2}{g} [y + ir \cos(\theta - a)] \right\}$

for large r.

The problem is divided into two parts, the homogeneous problem, with solution $X_H(r,\theta,y)$, satisfying (4.1), (4.2), (4.3) [with $f(r,y) \equiv 0$] and (4.4); and the inhomogeneous problem, with solution $X_I(r,\theta,y)$ satisfying (4.1), (4.2), (4.3) [with $f(r,y) \not\equiv 0$], and with $X_I(r,\theta,y) \longrightarrow 0$ as $r \longrightarrow \infty$. The solution $X(r,\theta,y)$ of the problem is then given by

$$X(r,\theta,y) = X_H(r,\theta,y) + X_I(r,\theta,y)$$

In the homogeneous problem, the techniques of Fourier transforms, separation of variables, and the use of various trigonometric and Bessel identities reduce the problem to solving a homogeneous difference equation, which is solved to yield

$$X_H(r,\theta,y) = \exp\left(\frac{\sigma^2 y}{g}\right) \sum_{n=0}^{\infty} \varepsilon_n \exp\left(\frac{in\pi}{4}\right) J_{\frac{n}{2}}\left(\frac{\sigma^2 r}{g}\right) \cos\frac{na}{2} \cos\frac{n\theta}{2}\,.$$

In the inhomogeneous case, the function $X_I(r,\theta,y)$ is written as the sum of two functions

$$X_I(r,\theta,y) = u(r,\theta,y) + v(r,\theta,y),$$

where $u(r,\theta,y)$ satisfies (4.1) and (4.3), and $u = 0$ on $y = 0$; so that $v(r,\theta,y)$ satisfies (4.1); $L[v] = -g\, u_y(r,\theta,0)$ on

$y = 0$; and $v_\theta = 0$ on $\theta = 0$ and on $\theta = 2\pi$.

The function $u(r,\theta,y)$ is found by constructing a Green's function

$$G(r_o,\theta_o,y_o;r,\theta,y) = \frac{1}{\pi}\sum_{n=0}^{\infty} \varepsilon_n \cos\frac{n\theta}{2}\cos\frac{n\theta_o}{2}\int_{-\infty}^{\infty}\exp[iq(y-y_o)]\cdot \int_0^{\infty}\frac{J_{\frac{n}{2}}(rp)J_{\frac{n}{2}}(r_o p)\,p\,dp\,dq}{p^2+q^2}$$

for the space bounded by the half plane $z = 0$, $x > 0$. Setting

$$G^*(r_o,y_o;r,\theta,y) = G(r_o,0,y_o;r,\theta,y) - G(r_o,2\pi,y_o;r,\theta,y)\ ,$$

and using Green's second identity determines $u(r,\theta,y)$.

$$u(r,\theta,y) = -\frac{1}{4\pi}\int_{-\infty}^{0}\int_{0}^{\infty}[G^*(r_o,y_o;r,\theta,y) - G^*(r_o,-y_o;r,\theta,y)]\cdot$$

$$f(r_o,y_o)\,dr_o\,dy_o\ .$$

The function $v(r,\theta,y)$ is found through the use of an auxilliary function

$$w(r,\theta,y) = L[v(r,\theta,y)],$$

where L is the differential operator given by (4.2). After $w(r,\theta,y)$ is found, the assumption that $v(r,\theta,y)$ has the same form as $w(r,\theta,y)$, the use of the relation $L[v] = w$, and the various trigonometric and Bessel identities used in the homogeneous case, together with the technique of Fourier transforms,

reduces the problem to an inhomogeneous difference equation, which is solved to yield

$$v(r,\theta,y) = \frac{g \cos\frac{\theta}{2}}{2\pi^3} \int_{-\infty}^{0} \int_{0}^{\infty} \int_{0}^{\infty} \int_{P'} \int_{P} \frac{f(r_o,y_o)(zz')^{-1/2}(1-zz')}{(zz')^2 - 2zz' \cos\theta + 1}$$

$$\frac{\exp\left[p(y + y_o) - \frac{rp}{2}(z' - \frac{1}{z'}) - \frac{r_o p}{2}(z - \frac{1}{z})\right]}{\frac{s_o^2 p^2}{4}(z - \frac{1}{z})^2 - is_o p\Omega(z - \frac{1}{z}) + (gp - \Omega^2)} \cdot$$

$$p\, dz'\, dz\, dp\, dr_o\, dy_o \cdot$$

where the path P is indicated in Figure 4, and P' is the same path in the z'-plane as P is in the z-plane.

Bibliography

1. Bateman, H. — Partial differential equations of mathematical physics, ch. 10, Cambridge, 1932.

2. Haskind, M. D. — The hydrodynamic theory of the oscillation of a ship in waves (in Russian), Prik. Mat. i. Mek., vol. 10, no. 1 (1946).

   Oscillation of a ship on a calm sea (in Russian), Izv. Ak. Nauk., Ot. Tex. Nauk., (1946).

   Translated by V. Wehausen as "Two papers on the hydrodynamic theory of heaving and pitching of a ship", Tech. Res. Bull. 1-12, Soc. Nav. Arch. and Mar. Eng., New York, (1953).

3. Hobson, E. W. — On Green's function for a circular disk, with applications to electrostatic problems, Cambr. Phil. Trans., 18 (1900).

4. John, F. — Waves in the presence of an inclined barrier, Comm. Pure and App. Math., vol. I, no. 2 (1948).

5. John, F. — On the motion of floating bodies I, II; Comm. Pure and App. Math., vol. II, no. 1 (1948), and vol. III, no. 2 (1950).

6. Lamb, H. — Hydrodynamics, Ch. 9, Dover, New York, 1945.

7. Neustadter, S. F. — Multiple valued harmonic functions with circle as branch curve, Univ. Calif. Publ. Math., N.S. 1 (1951).

8. Peters, A. S. — Water waves over sloping beaches and the solution of a mixed boundary value problem for $\nabla^2 \phi - k^2 \phi = 0$ in a sector, Comm. Pure and App. Math., vol. V, no. 1 (1952).

9. Peters, A. S. and Stoker, J. J. — A uniqueness theorem and a new solution for Sommerfeld's and other diffraction problems, Comm. Pure and App. Math., vol. VII, no. 3 (1954).

10. Stoker, J. J. and Peters, A. S. — The motion of a ship, as a floating rigid body in a seaway, N. Y. Univ., Inst. Math. Sci. 203 (1954).

11. Ursell, F. — The effect of a fixed vertical barrier on surface waves in deep water, Cambr. Phil. Proc. 43 (1947).

APPROVED DISTRIBUTION LIST

Chief of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 438 (3)

Commanding Officer
Office of Naval Research
Branch Office
The John Crerar Library Bldg.
86 East Randolph Street
Chicago 1, Illinois (1)

Commanding Officer
Office of Naval Research
Branch Office
346 Broadway
New York 13, New York (2)

Commanding Officer
Office of Naval Research
Branch Office
1030 East Green Street
Pasadena 1, California (1)

Commanding Officer
Office of Naval Research
Branch Office
1000 Geary Street
San Francisco 24, California (1)

Commanding Officer
Office of Naval Research
Navy #100, Fleet Post Office
New York, New York (5)

Director
Naval Research Laboratory
Washington 25, D. C.
Attn: Code 2021 (6)

Documents Service Center
Armed Services Technical
Information Agency
Knott Building
Dayton 2, Ohio (5)

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D. C.
Attn: Research Division (1)

Chief, Bureau of Ordnance
Department of the Navy
Washington 25, D. C.
Attn: Research and Development Division (1)

Office of Ordnance Research
Department of the Army
Washington 25, D. C. (1)

Commander
Air Research and Development Command
Office of Scientific Research
P. O. Box 1395
Baltimore 18, Maryland
Attn: Fluid Mechanics Division (1)

Director of Research
National Advisory Committee
for Aeronautics
1724 F Street, Northwest
Washington 25, D. C. (1)

Director
Langley Aeronautical Laboratory
National Advisory Committee
for Aeronautics
Langley Field
Virginia (1)

Director
National Bureau of Standards
Washington 25, D. C.
Attn: Fluid Mechanics Section (1)

Professor R. Courant
Institute of Mathematical
Sciences
New York University
25 Waverly Place
New York 3, New York (1)

Professor G. Kuerti
Department of Mechanical
Engineering
Case Institute of Technology
Cleveland, Ohio (1)

Professor W. R. Sears, Director
Graduate School of Aeronautical
Engineering
Cornell University
Ithaca, New York (1)

Chief, Bureau of Ships
Department of the Navy
Washington 25, D. C.
Attn: Research Division (1)
Code 420 Preliminary Design (1)

Chief of Naval Research
Department of the Navy
Washington 25, D. C.
Attn: Code 416 (1)
Code 460 (1)

Commander
Naval Ordnance Test Station
3202 E. Foothill Blvd.
Pasadena, California (1)

Commanding Officer and Director
David Taylor Model Basin
Washington 7, D. C.
Attn: Hydromechanics Lab. (1)
Hydrodynamics Div. (1)
Library (1)

California Institute of Technology
Hydrodynamic Laboratory
Pasadena 4, California (1)

Professor A. T. Ippen
Hydrodynamics Laboratory
Massachusetts Institute
Of Technology
Cambridge 39, Massachusetts (1)

Dr. Hunter Rouse, Director
Iowa Institute of Hydraulic
Research
State University of Iowa
Iowa City, Iowa (1)

Stevens Institute of Technology
Experimental Towing Tank
711 Hudson Street
Hoboken, New Jersey (1)

Dr. L. G. Straub
St. Anthony Falls Hydraulic
Laboratory
University of Minnesota
Minneapolis 14, Minnesota (1)

Dr. G. H. Hickox
Engineering Experiment Station
University of Tennessee
Knoxville, Tennessee (1)

Mr. C. A. Gongwer
Aerojet General Corporation
6352 N. Irwindale Avenue
Azusa, California (1)

Chief, Bureau of Yards and Docks
Department of the Navy
Washington 25, D. C.
Attn: Research Division (1)

Commanding Officer and Director
David Taylor Model Basin
Washington 7, D. C.
Attn: Ship Division (1)

Hydrographer
Department of the Navy
Washington 25, D. C. (1)

Director
Waterways Experiment Station
Box 631
Vicksburg, Mississippi (1)

Office of the Chief of Engineers
Department of the Army
Gravelly Point
Washington 25, D. C. (1)

Beach Erosion Board
U.S. Army Corps of Engineers
Washington 25, D. C. (1)

Commissioner
Bureau of Reclamation
Washington 25, D. C. (1)

Dr. G. H. Keulegan
National Hydraulic Laboratory
National Bureau of Standards
Washington 25, D. C. (1)

Brown University
Graduate Division of Applied
Mathematics
Providence 12, Rhode Island (1)

California Institute of Technology
Hydrodynamics Laboratory
Pasadena 4, California
Attn: Professor M. S. Plesset (1)
Professor V. A. Vanoni (1)

Professor M. L. Albertson
Department of Civil Engineering
Colorado A & M College
Fort Collins, Colorado (1)

Professor G. Birkhoff
Department of Mathematics
Harvard University
Cambridge 38, Massachusetts (1)

Massachusetts Institute of
Technology
Department of Naval Architecture
Cambridge 39, Massachusetts (1)

Dr. R. R. Revelle
Scripps Institute of Oceanography
La Jolla, California (1)

Stanford University
Applied Mathematics and
Statistics Laboratory
Stanford, California (1)

Professor J. W. Johnson
Fluid Mechanics Laboratory
University of California
Berkeley 4, California (1)

Professor H. A. Einstein
Department of Engineering
University of California
Berkeley 4, California (1)

Dean K. E. Schoenherr
College of Engineering
University of Notre Dame
Notre Dame, Indiana (1)

Director, Woods Hole Oceanographic
Institute
Woods Hole, Massachusetts (1)

Date Due

	PRINTED	IN U. S. A.	

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

1540

NYU
IMM
225 Jordan
AUTHOR
A mixed boundary potential problem arising from a problem in ship
TITLE

NYU
IMM
225 Jordan

A mixed boundary potential problem arising from a problem in ship motion.